GAUSSIAN PROCESSES
EXERCISE SHEET 9: GAUSSIAN FREE FIELD

Exercise 1 (Conditional density on a linear subspace)**.**

(1) Let X be a random vector in $\mathbb{R}^n$ with density f_X with respect to Lebesgue measure. Let $A \in \mathbb{R}^{k\times n}$ have full row rank, to study the conditional density of X on the affine subspace

$$S = \{x \in \mathbb{R}^n : Ax = b\},$$

we consider any invertible linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n, \qquad (y, z) = T(x),$$

with

$$y \in \mathbb{R}^k, \qquad z \in \mathbb{R}^{n-k},$$

and that

$$y = Ax.$$

Now let $(Y, Z) := T(X)$ and $f_{Z|Y=b}(z)$ be the conditional probability density of Z given $Y = b$. Show that $f_{Z|Y=b}(z)$ is proportional to $f_X(T^{-1}(b, z))$.

(2) Use (1) to prove that A is equivalent to B in the lecture last week.

Exercise 2 (Green's function on a graph)**.** *Consider the function G defined in part C for the lecture last week. Show that for any v, w*

$$d_v G(v, w) - \sum_{v'\sim v} G(v', w) = 1_{v=w}.$$

Exercise 3. *Let $G = (V, E)$ be the $(N + 1) \times (N + 1)$ square grid graph, so that $|V| = (N + 1)^2$. Let $V_\partial \subset V$ be its outer boundary.*

Let $g : E \to \mathbb{R}$ be a random function assigning to each edge $e \in E$ an independent standard Gaussian random variable $g(e)$, such that $g(e) = -g(-e)$. (In the lecture notes $g(e) = g(-e)$, which is a typo.)

Let $g_1 : E \to \mathbb{R}$ be the orthogonal projection of g onto the subspace described in Part B of last week's lecture.

Define $g_2 := g - g_1$. Show that g_2 has the same law as the gradient of a Gaussian Free Field on the dual graph G^* of G, where each vertex of G^* corresponds to a face of G, the boundary set is any one-point set, and two vertices of G^* are adjacent whenever the corresponding faces of G share an edge.

Exercise 4 (Not examinable)**.** *Consider Exercise 3 in higher dimensions, or on a general graph.*